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Twisted Cubics, bis

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Abstract. We consider the smooth compactification constructed in [12] for a space of varieties like twisted cubics. We show this compactification embeds naturally in a product of flag varieties.

Keywords: Hilbert schemes, rational curves, twisted cubics.

Introduction

Twisted cubics have long been a source of interesting examples and test cases in algebraic geometry. The Hilbert scheme $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ was studied by Piene and Schlessinger in [9], where we learn among other things that it consists of two smooth components intersecting transversally. The component parameterizing twisted cubics was investigated further by Ellingsrud, Piene and Strømme in [1], [2], [3]; see also [13]. For degree $d \geq 4$, the scheme $\mathrm{Hilb}^{dm+1}(\mathbb{P}^3)$ has more than two components, which are hard to describe; see Martin-Deschamps and Piene [8].

The interest in compactified parameter spaces for rational curves originated in enumerative questions considered by Schubert [11] and others; see Kleiman [6] for a survey. Recently many of these questions have been answered using stable maps; for example, see Fulton and Pandharipande [4] and Graber, Kock and Pandharipande [5]. Although techniques are still in short supply for varieties of higher dimension, particular cases are treated in [12]. For instance, that paper contains a determination of the number 648,151,945 of rational ruled cubic surfaces in $\mathbb{P}^4$ meeting 18 general lines, and a determination of the number 7,265,560,058,820 of Veronese surfaces in $\mathbb{P}^5$ meeting 24 general lines. (It also contains a determination of the number 15 of triangles in $\mathbb{P}^2$ meeting 6 general

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lines, and a determination of Schubert's number 80,160 of twisted cubics in $\mathbb{P}^3$ meeting 12 general lines.)

Section 1 reviews the construction in [12]. This construction produces a smooth compactification for varieties defined by nets of quadrics of determinantal type, such as twisted cubics, by an explicit sequence of three blowups starting from a nice variety. Lacking in [12], however, is a description of natural embeddings for the final two blowups. Although each center was well understood per se, it does not appear as the indeterminacy locus of a natural rational map. Besides being unaesthetic, this lack rendered the search for the closed orbits more technical than need be. These orbits are used to reduce the proof of the family's flatness to a simple computation.

The purpose of this note is to show that, in fact, this compactification embeds naturally into a product of flag varieties. This embedding lies in the long tradition of completing figures by adding geometrically meaningful aspects (see, for example, Kleiman and Thorup [7]). In the case of twisted cubics, we start with the Grassmannian parameterizing the pencils of quadrics with a common line. To a general pencil A, we attach a suitable net B of quadrics, designed to single out the twisted cubic residual to the line. Next we build a linear system C of eight independent cubics, and then we enlarge it to a 10-dimensional space D. Finally, we show the compactification is the closure of the locus of (A, B, C, D).

In Section 1, we introduce notation and review [12]. In Section 2, we state the main results, which identify the second blowup as the closure of the graph of the rational map $(A, B) \mapsto C$, and the third as that of the map $(A, B, C) \mapsto D$. In Section 3, we outline the proofs.

1 Quick review

For simplicity, we treat the case of twisted cubics with a fixed line L_0 as a chord. The general case is similar.

Denote by F the vector space of linear forms on the homogeneous coordinates $x_1, \ldots, x_4$. Our starting point is $\mathbb{X} = \mathbb{G}(2, 7)$, the Grassmannian of 2-dimensional subspaces of the vector space of quadrics vanishing on L_0. Each point in $\mathbb{X}$ is a pencil $A = \langle q_1, q_2 \rangle$ of quadrics containing L_0 in the base locus. For a general A, we get a residual twisted cubic $\mathfrak{t}_A$ meeting L_0 twice. This $\mathfrak{t}_A$ is the scheme of zeros of a net of quadrics $B = A + \langle q_3 \rangle$. Given linear forms x, y that define L_0, say $q_1 = ax + by$ and $q_2 = cx + dy$ with $a, b, c, d \in F$; then $ad - bc$ can be chosen as q_3

There is a rational map $\mathbb{X} \cdots \to \mathbb{G}(3, 10)$ with $A \mapsto B = A + \langle q_3 \rangle$. It fails to be a morphism precisely along the subscheme $\mathbb{Y}$ of $\mathbb{X}$ parameterizing the pencils of the form $\langle h \cdot h_1, h \cdot h_2 \rangle$ with $h \supset L_0$ a plane.

Blowing up $\mathbb{X}$ along $\mathbb{Y}$ yields a smooth subvariety $\mathbb{X}' \subset \mathbb{X} \times \mathbb{G}(3, 10)$. A general point in the fiber over $\langle h \cdot h_1, h \cdot h_2 \rangle$ represents the configuration of the line $\langle h_1, h_2 \rangle$ union a conic in the plane h.

Each point of $\mathbb{X}'$ corresponds to a flag $A \subset B$ such that B is a determinantal net and A is a pencil of quadrics. We have a vector bundle map defined fiberwise by multiplication $B \otimes F \to S_3 F$. It is shown in [12] that the Fitting subscheme $\mathbb{Y}'$ of $\mathbb{X}'$ where the rank drops to 9 consists of two smooth components, $\mathbb{Y}'_1$ and $\mathbb{Y}'_2$. The first is isomorphic to the incidence variety,

$$\mathbb{Y}'_1 \cong \{(p, h) \in \mathbb{P}^3 \times \check{\mathbb{P}}^3 \mid p \in h \cap L_0 \subset \mathbb{P}^3\}, \tag{1}$$

shown in Figure 1.

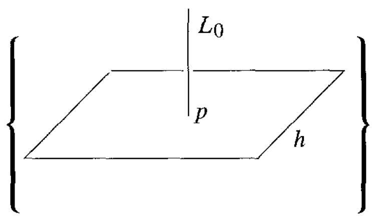

Figure 1: The incidence variety $\mathbb{Y}'_1$.

The second component is isomorphic to the incidence variety,

$$\mathbb{Y}'_2 \cong \{(p, \lambda, h) \in \mathbb{P}^3 \times \mathbb{G}(2, 4) \times \check{\mathbb{P}}^3 \mid p \in \lambda \subset h \supset L_0\},$$

shown in Figure 2.

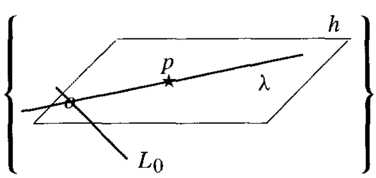

Figure 2: The incidence variety $\mathbb{Y}''_2$.

Let $\mathbb{X}''$ be the blowup of $\mathbb{Y}'_1 \subset \mathbb{X}'$, and $\mathbb{Y}''_2$ the strict transform of $\mathbb{Y}'_2$,

$$\mathbb{Y}''_2 \cong \{(p, o, \lambda, h) \in \mathbb{P}^3 \times L_0 \times \mathbb{G}(2, 4) \times \check{\mathbb{P}}^3 \mid p, o \in \lambda \subset h \supset L_0\} \cdot \tag{2}$$

Blowing up $\mathbb{Y}''_2 \subset \mathbb{X}''$ produces the smooth compactification $\mathbb{X}'''$ of [12].

2 Main results

The novelty reported on here is simply this: the blowup centers $\mathbb{Y}'_1$ and $\mathbb{Y}''_2$ appear as indeterminacy loci of natural rational maps.

First, the variety $\mathbb{Y}'_1$ appears singled out as an appropriate Fitting subscheme for the *saturation* of the image of the multiplication map from $A \otimes F$ (rather than $B \otimes F$!) into S_3F. The generic rank of $A \otimes F \to S_3F$ is 8, and drops to 7 precisely along $\mathbb{Y}'_1$. Note that its image $F \cdot A \subset S_3F$ actually lands into the 16-dimensional subspace $L_0 \cdot S_2F$ of cubic forms vanishing along the line L_0. We are abusing notation and identifying the line with the 2-dimensional space of linear forms that define it. We may state the following result.

Theorem 1. *Let $\mathbb{X}' \cdots\to \mathbb{G}(8, 16)$ be the rational map that assigns to a general flag (A, B) the vector space of cubic forms $F \cdot A \subset L_0 \cdot S_2F$. Then the blowup $\mathbb{X}''$ of $\mathbb{X}'$ along $\mathbb{Y}'_1$ embeds in $\mathbb{X}' \times \mathbb{G}(8, 16)$ as the closure of the graph of this rational map.*

Thus, the blowup $\mathbb{X}''$ of $\mathbb{X}'$ along $\mathbb{Y}'_1$ embeds as

$$\mathbb{X}'' \subset \mathbb{X}' \times \mathbb{G}(8, 16) \subset \mathbb{G}(2, 7) \times \mathbb{G}(3, 10) \times \mathbb{G}(8, 16). \tag{3}$$

It turns out that the strict transform $\mathbb{Y}''_2 \subset \mathbb{X}''$ of $\mathbb{Y}'_2$ is now the Fitting subscheme of the pullback of the map defined by $B \otimes F \to S_3F$, up to a saturation trick. Precisely, let $\mathcal{B}$ denote the pullback to $\mathbb{X}''$ of the tautological rank-3 subbundle of $S_2F_{|\mathbb{G}(3,10)}$. Let $\mathcal{D} \subset S_3F$ be the subsheaf obtained by saturation of the image of the multiplication map $\mathcal{B} \otimes F \to S_3F_{|\mathbb{X}''}$. The generic rank of $\mathcal{D}$ is 10 and the Fitting subscheme of $\mathbb{X}''$ where the rank of $S_3F/\mathcal{D}$ jumps is equal to $\mathbb{Y}''_2$. In other words, we may state the following result.

Theorem 2. *Let $\mathbb{X}'' \cdots\to \mathbb{G}(10, 20)$ be the rational map that assigns to a general point (A, B, C) the vector space of cubic forms $F \cdot B \subset S_3F$. Then the blowup $\mathbb{X}'''$ of $\mathbb{Y}''_2 \subset \mathbb{X}''$ embeds in $\mathbb{X}'' \times \mathbb{G}(10, 20)$ as the closure of the graph of this rational map.*

Thus, the blowup of $\mathbb{X}''$ along $\mathbb{Y}''_2$ embeds in the product of the two flag varieties, as stated in the introduction.

When [12] was accepted, its authors were unaware of this result. It was discovered during a related investigation [10] of the family of quintic curves of genus 2 in $\mathbb{P}^3$. The result simplifies the work in [12], especially the crucial determination of the closed orbits. Indeed, instead of having to deal with a tricky induced action on the normal bundle of $\mathbb{Y}'_1 \subset \mathbb{X}'$ (and with a similar one on that of $\mathbb{Y}''_2 \subset \mathbb{X}''$), we need only work out the natural induced actions on those Grassmann varieties, and this work is much simpler.

Under the action of the stabilizer of the the line $L_0 = \langle x_1, x_2 \rangle$, the relevant closed orbits turn out to have the following representatives:

$o = \langle x_1^2, x_1x_2 \rangle$ in $\mathbb{X} = \mathbb{G}(2, 7)$;

$o'_1 = (o, \langle x_1^2, x_1x_2, x_2^2 \rangle)$ in $\mathbb{X}' \subset \mathbb{X} \times \mathbb{G}(3, 10)$, which is a terminal orbit in the sense that it lies off the next blowup centers;

$o'_2 = (o, \langle x_1^2, x_1x_2, x_1x_3 \rangle)$ in $\mathbb{X}' \subset \mathbb{X} \times \mathbb{G}(3, 10)$;

$o''_2 = (o'_2, \ \langle x_1^2, x_1x_2 \rangle \cdot F + \langle x_1x_3^2 \rangle)$ in $\mathbb{X}'' \subset \mathbb{X}' \times \mathbb{G}(8, 16)$;

$o'''_2 = (o''_2, \ \langle x_1^2, x_1x_2, x_1x_3 \rangle \cdot F + \langle x_2^3 \rangle)$ in $\mathbb{X}''' \subset \mathbb{X}'' \times \mathbb{G}(10, 20)$.

3 The proofs

The proofs consist of a few simple local calculations as in [12]. Here is an outline. The two quadrics

$$\begin{cases} q_1 = x_1^2 + \qquad a_1x_1x_3 + a_2x_1x_4 + a_3x_2^2 + a_4x_2x_3 + a_5x_2x_4, \\ q_2 = \qquad x_1x_2 + a_6x_1x_3 + a_7x_1x_4 + a_8x_2^2 + a_9x_2x_3 + a_{10}x_2x_4 \end{cases}$$

give a local trivialization of the rank-2 tautological bundle in a neighborhood of $\langle x_1^2, x_1x_2 \rangle$ in $\mathbb{G}(2, 7)$. Writing $q_i = f_{i1}x_1 + f_{i2}x_2$ for suitable $f_{ij} \in F$, we may take $q_3 = \det(f_{ij})$. The rational map

$$\nu : \langle q_1, q_2 \rangle \mapsto \langle q_1, q_2, q_3 \rangle$$

can be locally represented by a 3×10 matrix with entries given by the coefficients of q_i with respect to the monomials $x_1^2, x_1x_2, \ldots, x_4^2$.

Performing row and column operations, we can bring the matrix into the triangular form $\begin{pmatrix} I_2 & \star \\ 0 & \alpha \end{pmatrix}$, with a 2×2 identity block. We can check that the indeterminacies of ν occur precisely along the subscheme $\mathbb{Y}$ locally defined by the entries in the row vector α. The local equations may be read amidst the entries of α,

$$a_9 = a_8a_6, \ a_{10} = a_8a_7, \ a_3 = -a_8^2, \ a_4 = a_1a_8, \ a_5 = a_2a_8.$$

Let $\mathbb{X}'$ be the blowup of $\mathbb{Y} \subset \mathbb{X}$. It is expressed in local coordinates by the assignments,

$$\begin{array}{ll} a_{10} = b_1\varepsilon_1 + a_8a_7, & a_3 = b_2\varepsilon_1 - a_8^2, \\ a_4 = b_3\varepsilon_1 + a_1a_8, & a_5 = b_4\varepsilon_1 + a_2a_8. \end{array} \tag{4}$$

Here the new local variables are $a_1, a_2, a_6, a_7, a_8, a_9, b_1, \dots, b_4$; furthermore, $\varepsilon_1 = a_9 - a_8 a_6$ is local generator of the exceptional ideal. This choice is made so that the origin is a representative of the closed orbit of $o_2' \in \mathbb{X}'$, which is acted on by the stabilizer of L_0.

Set for short $R = \mathbb{C}[a_1, \dots, b_4]$. Plugging the relations (4) into the above 3×10 matrix, we find the third row becomes divisible by ε_1. Dividing, we a find a matrix of rank 3 everywhere in the present neighborhood. The R-submodule B of $R^{10} \cong R \otimes S_2 F$ spanned by the three new rows is the saturation of the submodule B^0 spanned by the original rows. (By definition, the saturation B is the set of all q in the free module $R \otimes S_2 F$ such that $r \cdot q$ lands in B^0 for some nonzero $r \in R$.) Then B corresponds to a local trivialization q_1', q_2', q_3' for the rank-3 bundle inherited by $\mathbb{X}' \subset \mathbb{X} \times \mathbb{G}(3, 10)$. Note that B also comes equipped with the split submodule $A = \langle q_1', q_2' \rangle$, corresponding to the pencil.

In [12], we proceeded to compute a matrix representation for the multiplication map $B \otimes F \to R^{20} \cong R \otimes S_3 F$. Here, we look instead first at $A \otimes F \to R^{16} \cong R \otimes (L_0 \cdot S_2 F)$. Now the generic rank is 8. A matrix representation can be found in the triangular form $\binom{I_7\ \star}{0\ \beta}$. It turns out that the row vector β is a multiple of ε_1. Dividing, we find a local presentation for the saturation of the image. The new bottom row spans the ideal of the smooth subvariety $\mathbb{Y}_1'$ (see (1)) of codimension 7, with equations,

$$a_1 = a_9,\ a_2 = b_1 a_9,\ a_6 = 0,\ a_7 = 0,\ a_8 = -1/2 b_3,\ b_2 = 0,\ b_4 = b_1 b_3.$$

Blowing up $\mathbb{Y}_1' \subset \mathbb{X}'$ yields the embedding (3). In appropriate local coordinates, the map $\mathbb{X}'' \to \mathbb{X}'$ can be written thus:

$$\begin{array}{lll} a_7 = c_1 a_6, & b_2 = c_2 a_6, & a_8 = c_3 a_6 - 1/2 b_3, \\ b_4 = c_4 a_6 + b_1 b_3, & a_1 = c_5 a_6 + a_9, & a_2 = c_6 a_6 + b_1 a_9. \end{array}$$

Now $\varepsilon_2 = a_6$ gives the exceptional divisor.

As before, we substitute the above relations into our 8×16 matrix, and divide the bottom row by ε_2. Next, we enlarge the matrix to size 12×20 by first putting four zero columns corresponding to cubic monomials involving only x_3, x_4, then four new rows corresponding to the coefficients of the products $q_3 \cdot x_i$ for $i = 1, \dots, 4$. The resulting matrix represents the multiplication map $B \otimes F \to R^{20}$. Its minimal rank is 9.

Performing elementary row and column operations and discarding zero rows, we can bring the matrix into triangular form $\binom{I_9\ \star}{0\ \gamma}$. Here the row vector γ generates the ideal of the smooth subvariety $\mathbb{Y}_2''$ (see (2)) of codimension 6, locally given by

$$c_2 = c_3 = c_4 = 0,\ b_3 = 2c_5/3,\ c_6 = (2c_1 + b_1)c_5/3,\ a_9 = -c_5 a_6/3.$$

The pictures above, as well as a hint of the global descriptions of $\mathbb{Y}'_1$, $\mathbb{Y}''_2$, are found by substituting the corresponding local equations into the three generators for the net of quadrics and into the system of eight cubics.

There is a short script for `maple` with all the computational details in [14].

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